

An $SO(10)$ Solution to the Puzzle of Quark and Lepton Masses *

K.S.Babu and S.M.Barr

Bartol Research Institute
University of Delaware
Newark, DE 19716

BA-95-11, hep-ph/9503215

Abstract

It is shown that almost all features of the quark and lepton masses can be satisfactorily and simply explained without family symmetry, including the threefold mass hierarchy among the generations, and the relations $m_\tau^0 \cong m_b^0$, $m_\mu^0 \cong 3m_s^0$, $m_e^0 \cong \frac{1}{3}m_d^0$, $m_u^0/m_t^0 \ll m_d^0/m_b^0$, $\tan \theta_c \cong \sqrt{m_d^0/m_s^0}$, $V_{cb} \ll \sqrt{m_s^0/m_b^0}$, and $V_{ub} \sim V_{cb}V_{us}$. Various aspects of the group theory of $SO(10)$ play an essential role in explaining these relations. The form of the mass matrices, rather than being imposed arbitrarily, emerges naturally from a simple structure at the unification scale. This structure involves only vector, spinor and adjoint representations. There are distinctive and testable predictions for $\tan \beta$ and the neutrino mixing angles.

*Supported in part by Department of Energy Grant #DE-FG02-91ER406267

In this Letter we show that the group theory of $SO(10)$ can provide a satisfactory explanation of almost all of the features of the quark and lepton masses and mixing angles¹ without family symmetry. We propose a model, in two versions, each of which has several testable consequences for neutrino mixing angles and for the parameter $\tan \beta$.

We propose that the pattern of fermion masses is determined by the following Yukawa terms in the superpotential².

$$\begin{aligned}
W = & M(\overline{\mathbf{16}} \mathbf{16}) + \sum_{i=1}^3 b_i(\mathbf{16}_i \overline{\mathbf{16}}) \mathbf{45}_H + \sum_{i=1}^3 a_i(\mathbf{16}_i \mathbf{16}) \mathbf{10}_H \\
& + d(\mathbf{10} \mathbf{10}') \mathbf{45}_H^{(X)} + \sum_{i=1}^3 c_i(\mathbf{16}_i \mathbf{10}) \mathbf{16}_H + \sum_{i=1}^3 c'_i(\mathbf{16}_i \mathbf{10}') \mathbf{16}_H.
\end{aligned} \tag{1}$$

In addition to the three ordinary families of quarks and leptons, which are contained in the spinors denoted $\mathbf{16}_i$, there is a pair of spinor and antispinor $(\mathbf{16}, \overline{\mathbf{16}})$, and a pair of vectors $(\mathbf{10}, \mathbf{10}')$. The Higgs supermultiplets are distinguished by the subscript ‘ H ’. We will henceforth call the first three terms of eq.(1) W_{spinor} and the last three terms W_{vector} .

It will be shown that this simple structure explains the following nine well-known features of the quark and lepton spectrum.

The hierarchy:

(I) The first generation is much lighter than the second and third.

(II) The second generation is much lighter than the third.

The mass ratios:

(III) $m_b^0 \cong m_\tau^0$ (see Ref. 3). [The superscript ⁰ refers throughout to quantities at the unification scale.]

(IV) $|m_\mu^0/m_s^0| \cong 3$.

(V) $|m_e^0/m_d^0| \cong |m_\mu^0/m_s^0|^{-1} \cong \frac{1}{3}$. (The Georgi-Jarlskog relations.⁴)

(VI) $m_u^0/m_t^0 \ll m_d^0/m_b^0$ and m_e^0/m_τ^0 . (10^{-5} versus respectively 10^{-3} and 0.3×10^{-3} .)

The mixing angles:

(VII) V_{cb} is small compared to $\sqrt{m_s^0/m_b^0}$.

(VIII) $\tan \theta_c \cong \sqrt{m_d^0/m_s^0}$ (see Ref. 5).

(IX) $V_{ub} \sim V_{cb}V_{us}$.

In understanding the structure of eq.(1) a crucial point is that there is no Yukawa term $\sum_{i,j=1}^3 f_{ij}(\mathbf{16}_i \mathbf{16}_j) \mathbf{10}_H$ coupling the ordinary families to each other⁶, for the coefficient of such a term, f_{ij} , would be a matrix in ‘family space’, and one would naturally expect the three eigenvalues of such a matrix to be all of the same order rather than to exhibit a hierarchical pattern as observed in nature. Instead, what happens in this model is that the families ($\mathbf{16}_i$) couple to each of the ‘extra’ fermion multiplets ($\mathbf{16}, \overline{\mathbf{16}}, \mathbf{10}, \mathbf{10}'$) with Yukawa couplings that are *vectors* in family space (a_i, b_i, c_i, c'_i). This leads, to first approximation, as will be seen, to a ‘factorized’ form⁷ for the mass matrices, $M_{ij} \sim a_i b_j$, which because it has rank less than three does produce a hierarchy.

Another role played by the ‘extra’ fermions is to allow the $\mathbf{45}_H$, which breaks $SO(10)$, to couple directly to the quarks and leptons as it could not renormalizably do were there only the ordinary families, $\mathbf{16}_i$. This is important because some of the predictions of minimal $SO(10)$, such as $m_\mu^0 = m_s^0$ and $m_c^0/m_t^0 = m_s^0/m_b^0$, are badly broken in nature.

We will also discuss a variant of this model which is obtained by adding

to the superpotential of eq.(1) the following piece.

$$W_{adjoint} = f(\mathbf{45} \mathbf{45}') \mathbf{45}_H + \sum_{i=1}^3 e_i(\mathbf{16}_i \mathbf{45}) \overline{\mathbf{16}}_H + \sum_{i=1}^3 e'_i(\mathbf{16}_i \mathbf{45}') \overline{\mathbf{16}}_H. \quad (2)$$

The structure of this set of terms is quite analogous to W_{spinor} and W_{vector} . Here the ‘extra’ fermions are adjoints $(\mathbf{45}, \mathbf{45}')$ which couple, as before, to the ordinary families with coefficients which are vectors in family space (e_i, e'_i) . Note that because the Higgs fields $\mathbf{45}_H^{(X)}$ and $\mathbf{45}_H$ are antisymmetric tensors, the $\mathbf{10}$ and $\mathbf{10}'$ in W_{vector} must be distinct, as must the $\mathbf{45}$ and $\mathbf{45}'$ in $W_{adjoint}$.

We will denote the mass matrices of the quarks and leptons by U , D , L , N , and M_R . More precisely, $W_{mass} = \sum_{i,j=1}^3 [u_{Li}^c U_{ij} u_{Lj} + d_{Li}^c D_{ij} d_{Lj} + l_{Li}^+ L_{ij} l_{Lj}^- + \nu_{Li}^c N_{ij} \nu_{Lj} + \nu_{Li}^c (M_R)_{ij} \nu_{Lj}^c]$. The dominant contributions to these matrices are assumed to come from W_{spinor} and have the exact form⁸ (up to terms of order $M_W/M_{GUT} \sim 10^{-14}$)

$$U_0 = aT v \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & Q_u \sin \theta / N_u \\ 0 & Q_{u^c} \sin \theta / N_{u^c} & (Q_{u^c} + Q_u) \cos \theta / N_{u^c} N_u \end{pmatrix}, \quad (3)$$

$$D_0 = aT v' \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & Q_d \sin \theta / N_d \\ 0 & Q_{d^c} \sin \theta / N_{d^c} & (Q_{d^c} + Q_d) \cos \theta / N_{d^c} N_d \end{pmatrix}, \quad (4)$$

$$L_0 = aT v' \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & Q_{l^-} \sin \theta / N_{l^-} \\ 0 & Q_{l^+} \sin \theta / N_{l^+} & (Q_{l^+} + Q_{l^-}) \cos \theta / N_{l^+} N_{l^-} \end{pmatrix}, \quad (5)$$

$$N_0 = aT v \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & Q_\nu \sin \theta / N_\nu \\ 0 & Q_{\nu^c} \sin \theta / N_{\nu^c} & (Q_{\nu^c} + Q_\nu) \cos \theta / N_{\nu^c} N_\nu \end{pmatrix}. \quad (6)$$

Here θ is the angle between the Yukawa coupling constant vectors a_i and b_i appearing in eq.(1), and a and b are their lengths. Q is an $SO(10)$ generator⁹

giving the direction in group space of the vacuum expectation value (VEV) of the adjoint Higgs field, $\mathbf{45}_H$,

$$\langle \mathbf{45}_H \rangle = \Omega Q, \quad (7)$$

and Q_f is the Q charge of the fermion f . Ω , like M , is of order M_{GUT} , so that we define a dimensionless ratio $b\Omega/M \equiv T$ which is of order unity. The N_f are defined by $N_f \equiv \sqrt{1 + T^2 |Q_f|^2}$. (These factors of N_f do not play a significant role in what follows. For $T < 1$ they are close to one. We keep them for the sake of exactness.) Finally, v and v' are the usual $SU(2)_L \times U(1)_Y$ -breaking VEVs of the $\mathbf{5}(\mathbf{10}_H)$ and $\bar{\mathbf{5}}(\mathbf{10}_H)$ respectively. (Throughout ‘ $\mathbf{p}(\mathbf{q})$ ’ denotes a \mathbf{p} of $SU(5)$ contained in a \mathbf{q} of $SO(10)$.)

The exact forms given in eqs.(3) – (6) can be simply derived by straightforward algebra, but can be more easily understood from Fig. 1.

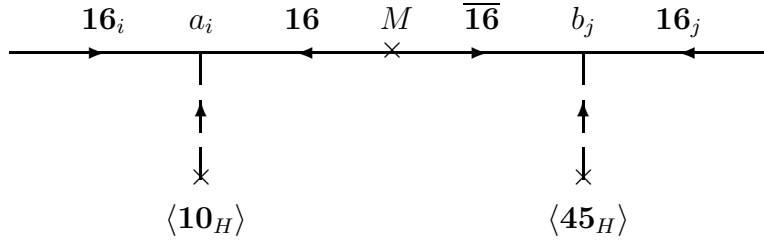


Fig. 1

By inspection, that diagram gives the expression

$$W_{mass} = \sum_{i,j=1}^3 a_i b_j \frac{\langle \mathbf{10}_H \rangle \langle \mathbf{45}_H \rangle}{M} (\mathbf{16}_i \mathbf{16}_j); \quad (8)$$

or $W_{mass} = aT \langle \mathbf{10}_H \rangle (\sum_{i=1}^3 \hat{a}_i \mathbf{16}_i) (\sum_{j=1}^3 \hat{b}_j Q_{16_j} \mathbf{16}_j)$, where $\hat{a}_i \equiv a_i/a$, $\hat{b}_i \equiv b_i/b$. For the up quarks this gives

$$W_{mass}^{up} = aTv \sum_{i,j=1}^3 [\hat{a}_i \hat{b}_j Q_u + \hat{a}_j \hat{b}_i Q_{u^c}] u_i^c u_j, \quad (9)$$

with similar expressions for the down quarks and leptons. Without any loss of generality one may choose the axes in family space so that

$$\begin{aligned} \hat{a}_i &= (0, \sin \theta, \cos \theta), \\ \hat{b}_i &= (0, 0, 1), \end{aligned} \quad (10)$$

which when substituted into eq.(9) and its analogues gives the matrices displayed in eqs.(3) – (6) with $N_f = 1$. (To build up the factors N_f^{-1} one must sum over all tree graphs with arbitrary numbers of superheavy mass or VEV insertions. More simply one can just do the algebra of integrating out the $\mathbf{16}$ and $\overline{\mathbf{16}}$.)

Because $\langle \mathbf{45}_H \rangle$ cannot break the Standard Model gauge group, Q must be a linear combination of the hypercharge and the $SU(5)$ -singlet generator, X . ($SO(10) \supset SU(5) \times U(1)_X$.) For convenience Q will be normalized so that

$$Q = \left(\frac{-1}{5}\right)X + \left(\frac{6}{5}z\right)\frac{Y}{2} = 2 I_{3R} + \left(\frac{6}{5}\epsilon\right) \frac{Y}{2}, \quad (11)$$

where X is normalized conventionally, so that $X_{\mathbf{10}(\mathbf{16})} = 1$, $X_{\overline{\mathbf{5}}(\mathbf{16})} = -3$, $X_{\mathbf{1}(\mathbf{16})} = 5$; and $\epsilon \equiv z - 1$. Thus, for the fermions contained in the $\mathbf{16}$ of

$SO(10)$,

$$\begin{aligned}
Q_u &= Q_d = \frac{1}{5}\epsilon \\
Q_{u^c} &= -1 - \frac{4}{5}\epsilon \\
Q_{d^c} &= 1 + \frac{2}{5}\epsilon \\
Q_{l^-} &= Q_\nu = -\frac{3}{5}\epsilon \\
Q_{l^+} &= 1 + \frac{6}{5}\epsilon \\
Q_{\nu^c} &= -1.
\end{aligned} \tag{12}$$

We shall assume, for reasons that will become apparent shortly, that Q is oriented approximately in the I_{3R} direction; that is, that $|\epsilon| \ll 1$. Then it is seen from their definitions that the N_f for all the left-handed fermions (u, d, l^-, ν) are very close to one, while for all the left-handed anti-fermions (u^c, d^c, l^+, ν^c) $N_f \cong \sqrt{1+T^2} \equiv N$.

Several striking features⁸ of the mass matrices in eqs.(3) – (6) are evident upon inspection.

As a consequence of factorization (cf. eq.(9)) the mass matrices are rank 2. This explains **(I)**, the lightness of the first generation. Also explained is **(III)**, $m_b^0 \cong m_\tau^0$. The origin of this relation is the fact that $D_{33} \cong aTv'(Q_{d^c} + Q_d) \cos \theta / N = aTv'(Q_{l^+} + Q_{l^-}) \cos \theta / N \cong L_{33}$. The equality of these expressions is no accident, but a consequence of the fact that $l^+ l^-$ and $d^c d$ both couple to the same Higgs doublet, ϕ' , and thus must have equal charges. On the other hand, eqs.(4) and (5) show that $|m_\mu^0/m_s^0| \neq 1$ but rather

$$\left| \frac{m_\mu^0}{m_s^0} \right| \cong \left| \frac{m_\mu^0 m_\tau^0}{m_s^0 m_b^0} \right| = \left| \frac{\det_{23} L}{\det_{23} D} \right| \cong \left| \frac{Q_{l^+} Q_{l^-}}{Q_{d^c} Q_d} \right| = 3 \left| \frac{1 + 6\epsilon/5}{1 + 2\epsilon/5} \right|. \tag{13}$$

Thus the empirical relation **(IV)**, $m_\mu^0/m_s^0 \cong 3$ is seen to follow if Q is approximately in the I_{3R} direction; that is, if $|\epsilon| \ll 1$. (Cf. eq.(11).)

The I_{3R} direction is a natural one for the vacuum expectation value of a

Higgs field in the adjoint representation of $SO(10)$. For example,¹⁰ the superpotential $-\mu tr(A^2)+\lambda tr(A^4)$, where A is a **45**, only has $SU(3)\times SU(2)\times U(1)$ -invariant solutions in the I_{3R} , $B-L$, and X directions. Other interesting superpotentials¹¹ give the same possibilities. As evidence that adjoint Higgs might well have potentials of this type, it is significant that the Dimopoulos-Wilczek mechanism¹², which is necessary to solve the ‘doublet-triplet splitting problem’ in $SO(10)$ requires the existence of an adjoint Higgs whose vacuum expectation value is in the $B-L$ direction.

The assumption that Q is approximately in the I_{3R} direction would also provide a group-theoretical explanation for two other facts, **(II)** and **(VII)**. **(II)**, that the second generation is much lighter than the third, can be seen from eqs.(3) – (6), which reveal that the (2,3) elements of all the matrices vanish as $\epsilon \rightarrow 0$, causing the matrices to become rank 1. For example, $m_s^0/m_b^0 \cong \frac{N}{5}\epsilon \sin^2 \theta$. That **(VII)**, V_{cb}^0 is small compared to $\sqrt{m_s^0/m_b^0}$, follows from $V_{cb}^0 \cong \tan^{-1}(U_{32}/U_{33}) - \tan^{-1}(D_{32}/D_{33}) \cong \frac{2}{5}\epsilon \sin \theta \cos \theta$. In the $\epsilon \rightarrow 0$ limit both V_{cb}^0 and $V_{cb}^0/\sqrt{m_s^0/m_b^0} \cong \sqrt{4\epsilon/5N} \cos \theta$ vanish. It is well-known that Fritzsch-type relations¹³ for V_{cb}^0 such as $|V_{cb}^0| = |\sqrt{m_s^0/m_b^0} - e^{i\alpha} \sqrt{m_c^0/m_t^0}|$ fail even though the analogous relation for the Cabibbo angle works well, because the observed V_{cb}^0 is about a factor of three too small. Here that smallness is explained as due to the smallness of ϵ .

There is one potentially troubling feature of the forms given in eqs.(3) – (6) and that is that in the small ϵ limit the bad ‘proportionality’ prediction of naive $SO(10)$, namely $m_c^0/m_t^0 \cong m_s^0/m_b^0$, holds. The violation of this relation will be explained in an interesting way below.

Taking into account the heretofore neglected terms W_{vector} one can show

that the exact mass matrices (up to terms of order $M_W/M_{GUT} \sim 10^{-14}$) coming from eq.(1) are (using eq.(12))

$$U = U_0 = aTv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{5}\epsilon \sin \theta / N_u \\ 0 & -(1 + \frac{4}{5}\epsilon) \sin \theta / N_{u^c} & -(1 + \frac{3}{5}\epsilon) \cos \theta / N_u N_{u^c} \end{pmatrix}, \quad (14)$$

$$D = aTv'(I + \Delta_{d^c})^{-\frac{1}{2}} \begin{pmatrix} 0 & -c_{12} & -c_{13}/N_d \\ c_{12} & 0 & (\frac{1}{5}\epsilon \sin \theta - c_{23})/N_d \\ c_{13}/N_{d^c} & ((1 + \frac{2}{5}\epsilon) \sin \theta + c_{23})/N_{d^c} & (1 + \frac{3}{5}\epsilon) \cos \theta / N_d N_{d^c} \end{pmatrix}, \quad (15)$$

$$L = aTv' \begin{pmatrix} 0 & c_{12} & c_{13}/N_{l^-} \\ -c_{12} & 0 & (-\frac{3}{5}\epsilon \sin \theta + c_{23})/N_{l^-} \\ -c_{13}/N_{l^+} & ((1 + \frac{6}{5}\epsilon) \sin \theta - c_{23})/N_{l^+} & (1 + \frac{3}{5}\epsilon) \cos \theta / N_{l^+} N_{l^-} \end{pmatrix} (I + \Delta_{l^-}^T)^{-\frac{1}{2}}, \quad (16)$$

$$N = N_0 = aTv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{3}{5}\epsilon \sin \theta / N_\nu \\ 0 & -\sin \theta / N_{\nu^c} & -(1 + \frac{3}{5}\epsilon) \cos \theta / N_{\nu^c} N_\nu \end{pmatrix}. \quad (17)$$

One sees that U_0 and N_0 are unaffected by adding W_{vector} and that the effect on D_0 and L_0 is to add to them an antisymmetric piece, c_{ij} , and to multiply them by mixing matrices $(I + \Delta)^{-\frac{1}{2}}$. For the moment we will assume that these factors can be neglected as would be the case if $\Delta_{d^c}, \Delta_{l^-} \ll 1$, although such matrices will play an important role in our later discussion. (Their exact forms are given by $(\Delta_{d^c})_{ij} = \left| \frac{\langle \mathbf{1(16)}_H \rangle}{d\langle \mathbf{45}^{(X)}_H \rangle} \right|^2 p_i p_j (c_i^* c_j + c_i'^* c_j')$ where $\vec{p} \equiv (1, 1, N_{d^c}^{-1})$ and the same for Δ_{l^-} with N_{d^c} replaced by N_{l^-} .) The origin of the antisymmetric pieces, c_{ij} , can be understood from Fig. 2.

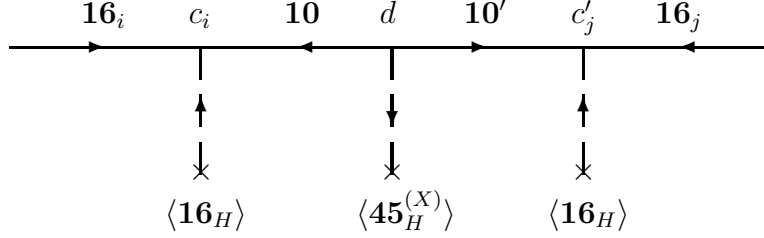


Fig. 2

By inspection of that diagram one sees that W_{vector} contributes

$$\Delta W_{mass} = \sum_{i,j=1}^3 (c_i c'_j - c'_i c_j) \frac{\langle \mathbf{1}(\mathbf{16}_H) \rangle \langle \bar{\mathbf{5}}(\mathbf{16}_H) \rangle}{d \langle \mathbf{45}_H^{(X)} \rangle} \mathbf{10}(\mathbf{16}_i) \bar{\mathbf{5}}(\mathbf{16}_j). \quad (18)$$

The antisymmetry in flavor is due to the antisymmetry of the adjoint of $SO(10)$. Under the interchange of $\mathbf{10}$ and $\mathbf{10}'$ in Fig.2 the diagram changes sign and $ij \rightarrow ji$. Comparing with eqs.(15) and (16) one obtains $c_{ij} = (c_i c'_j - c'_i c_j) \frac{M \langle \mathbf{1}(\mathbf{16}_H) \rangle}{abd \langle \mathbf{45}_H \rangle \langle \mathbf{45}_H^{(X)} \rangle} \frac{\langle \bar{\mathbf{5}}(\mathbf{16}_H) \rangle}{\langle \bar{\mathbf{5}}(\mathbf{10}_H) \rangle}$. We assume that the vacuum expectation value of the $\mathbf{45}_H^{(X)}$ is in the X direction so that the antisymmetric pieces added to D and L are the same. The smallness of the c_{ij} can come from the smallness of the c_i or c'_i , or of their cross product, or of the ratios of vacuum expectation values.

The forms of the full matrices exhibited in eqs.(14) – (17) now explain the several further relationships (V), (VI), and (VIII). Relation (V), that $|m_e^0/m_d^0| \cong |m_s^0 m_b^0/m_\mu^0 m_\tau^0| \cong 1/3$, is a consequence of $\det D = \det L$, which,

it should be noted, is an *exact* relation for any values of the parameters both in the limit $\Delta \rightarrow 0$ and in the limit $N_f \rightarrow 1$ (which makes $\Delta_{d^c} = \Delta_{l^-}$).

Relation **(VI)**, that m_u^0 is proportionately very tiny compared to m_d^0 and m_e^0 , comes from the fact that W_{vector} contributes only to D and L and leaves U rank 2, and thus u massless. Some higher order effects presumably give a small mass to u , but as these contributions to U would typically be of order $10^{-5}aTv$ one would expect them to have a negligible effect on everything else besides m_u . (For example, their contribution to the Cabibbo angle would typically be of order $m_u/m_c \sim 0.005$.)

Relation **(VIII)**, $\tan \theta_c \cong \sqrt{m_d^0/m_s^0}$ is famously successful.⁵ In schemes of the Fritzsch type¹³ there is an extra contribution $e^{i\alpha} \sqrt{m_u^0/m_c^0}$, which, being of magnitude $0.07 \cong \frac{1}{3} \tan \theta_c$ requires that the phase α take a particular value to obtain numerical agreement. Here, because W_{vector} does not contribute to U , one avoids that extra term.

The foregoing three relations are consequences of the facts that W_{vector} only contributes to D and L (required for **(VI)** and **(VIII)**) and that its contribution is antisymmetric (required for **(V)** and **(VIII)**). But it should be emphasized that these facts in turn are consequences of aspects of $SO(10)$ group theory: namely, that the vector representation contains only down quarks and leptons, and that the adjoint representation is antisymmetric.

The final relation, **(IX)**, $V_{ub} \sim V_{cb}V_{us}$, follows if one makes the natural assumption that all the c_{ij} are of the same order, since $V_{ub} \sim c_{13}/m_b^0$, $V_{us} \sim c_{12}/m_s^0$, and $V_{cb} \sim m_s^0/m_b^0$. This assumption also would imply that c_{23} plays a negligible role. So far, then, there are seven relevant combinations of parameters¹⁴: aTv , v/v' , $\sin \theta$, ϵ , c_{12} , c_{13} , and N . (And for small T , $N \cong 1$

and plays no role.)

Now comes the question of why $(m_c^0/m_t^0)/(m_s^0/m_b^0) \lesssim \frac{1}{5}$, when according to naive $SO(10)$ and eqs.(3), (4), and (12) it ought to be approximately equal to unity. There are two possible simple answers, that m_b^0 or m_c^0 is suppressed, and they lead to the two versions of the model referred to earlier. The first version, where m_b^0 is suppressed, has greater economy as it makes do with only the terms in eq.(1). In this version the factor $(I + \Delta_{d^c})^{-\frac{1}{2}}$ which multiplies D_0 on the left in eq.(15) is assumed to have the approximate form $\text{diag}(1, 1, \delta)$ where $\delta \lesssim \frac{1}{5}$. Then $m_b^0 \cong (D_{32}^2 + D_{33}^2)^{\frac{1}{2}}$ gets multiplied by δ , while $m_s^0 \cong D_{23}D_{32}/D_{33}$ is left unaffected. While this works, it turns out to have some minor drawbacks. The angle θ ends up being somewhat small, and, furthermore, for $(I + \Delta_{d^c})^{-\frac{1}{2}}$ to have the desired form the vector c_i (or alternatively c'_i) must also be nearly aligned with b_i . Such a preferred direction in family space is somewhat unappealing since family symmetry has been eschewed. A second drawback is that $m_b^0 \cong m_\tau^0$ is no longer automatic but must be fit. Aside from this, this version of the model preserves the explanatory successes **(I) – (IX)** and has in addition several interesting predictions.¹⁵

$$\begin{aligned} \tan\beta &\cong v/v' \cong m_c^0/m_s^0, \\ |\theta_{e\mu}| &\cong \sqrt{m_e^0/m_\mu^0} + \frac{1}{2} \tan\theta [Re(\theta_{e\tau})/(1 + |\theta_{e\tau}|^2)], \\ \sin^2 2\theta_{\mu\tau} &\gtrsim \frac{1}{2}. \end{aligned} \quad (19)$$

Note that since the ratio m_t^0/m_b^0 has been affected by the factor $(I + \Delta_{d^c})^{-\frac{1}{2}}$, $\tan\beta$ is given by the (unaffected) ratio m_c^0/m_s^0 , which is quite a distinctive prediction. The largeness of $\theta_{\mu\tau}$ arises because L_{32}/L_{33} is no longer $\cong \tan\theta$ but (a careful calculation shows) $\cong \frac{1}{2}\delta^{-1} \tan\theta$, whereas $N_{32}/N_{33} \sim \tan\theta$.

We shall now discuss the second and in some ways cleaner version of the model, in which m_c^0 is suppressed. We shall see that if the terms $W_{adjoint}$ given in eq.(2) are added to the superpotential then the ratio m_c^0/m_t^0 is typically suppressed relative to m_s^0/m_b^0 by a factor of $O(\epsilon)$ *without any special alignment of vectors in family space or special values of parameters*. In other words, the smallness of m_c^0/m_t^0 has a group-theoretical origin. This suppression is achieved by a matrix $(I + \Delta_{u^c})^{-\frac{1}{2}}$ appearing in eq.(14) analogous to the matrices appearing in eqs.(15) and (16).

Since $\langle \mathbf{1}(\overline{\mathbf{16}}_H) \rangle \neq 0$ and is $O(M_{GUT})$, $W_{adjoint}$ causes the $\mathbf{10}(\mathbf{16}_i)$ to mix with the $\mathbf{10}(\mathbf{45})$ so: $\overline{\mathbf{10}}(\mathbf{45}') \left[f\Omega Q_{\mathbf{10}(\mathbf{45})} \mathbf{10}(\mathbf{45}) + \langle \mathbf{1}(\overline{\mathbf{16}}_H) \rangle e'_i \mathbf{10}(\mathbf{16}_i) \right]$, with similar mixing of $\mathbf{10}(\mathbf{16}_i)$ and $\mathbf{10}(\mathbf{45}')$ caused by the e_i term. In particular, since the $\mathbf{10}(\mathbf{16}_i)$ contains the u_i^c there will be produced in eq.(14) a mixing matrix $(I + \Delta_{u^c})^{-\frac{1}{2}}$ multiplying U_0 on the left, with Δ_{u^c} given by

$$(\Delta_{u^c})_{ij} = \left| \frac{\langle \mathbf{1}(\overline{\mathbf{16}}_H) \rangle}{f\Omega Q_{u^c(\mathbf{45})}} \right|^2 \left[E_i^* E_j + E_i'^* E_j' \right], \quad (20)$$

where $E_i \equiv p_i e_i$, $E_i' \equiv p_i e'_i$, and $p_i \equiv (1, 1, N_{u^c}^{-1})$. (Compare with the expression for Δ_{d^c} given earlier.) Since the $\mathbf{10}(\mathbf{16}_i)$ also contains the u_i , d_i , and l_i^+ , there will be analogous mixing matrices $(I + \Delta_u)^{-\frac{1}{2}}$, $(I + \Delta_d)^{-\frac{1}{2}}$, and $(I + \Delta_{l^+})^{-\frac{1}{2}}$ introduced in the obvious places in eqs.(14) – (16), where Δ_u , Δ_d , and Δ_{l^+} are given by eq.(20) with $Q_{u^c(\mathbf{45})}$ being replaced by $Q_{u(\mathbf{45})}$, $Q_{d(\mathbf{45})}$, and $Q_{l^+(\mathbf{45})}$, respectively, and N_{u^c} replaced by the appropriate N 's.

What is required to suppress m_c^0/m_t^0 is that at least some elements of Δ_{u^c} should be quite large, whereas to preserve the successful relations (I) – (IX) the elements of Δ_u , Δ_d , and Δ_{l^+} should be somewhat (but not necessarily very much) smaller than unity. The required largeness of Δ_{u^c} is explained

rather elegantly by group theory. The crucial point is that the $Q_{u^c(\mathbf{45})}$ appearing squared in the denominator of eq.(20) is *not* the Q charge given in eq.(12) of the u_i^c that is contained in the $\mathbf{16}_i$. Rather, it is the Q charge of the u^c that are contained in the $\mathbf{45}$ and $\mathbf{45}'$. The difference is that $X_{\mathbf{10}(\mathbf{45})} = -4$, whereas $X_{\mathbf{10}(\mathbf{16})} = 1$, so that by eq.(11) $Q_{\mathbf{10}(\mathbf{45})} = Q_{\mathbf{10}(\mathbf{16})} + 1$. From eq.(12), then, one sees that $Q_{u^c(\mathbf{45})} = -\frac{4}{5}\epsilon$, and therefore $\Delta_{u^c} \propto (\frac{1}{\epsilon})^2$. The expressions for Δ_u , Δ_d , and Δ_{l^+} , on the other hand, have in their denominators $Q_{u(\mathbf{45})}^2 = Q_{d(\mathbf{45})}^2 = (1 + \frac{1}{5}\epsilon)^2$ and $Q_{l^+(\mathbf{45})}^2 = (2 + \frac{6}{5}\epsilon)^2$ and consequently are not enhanced.

It might be thought that some special form of Δ_{u^c} might have to be assumed to get a suppression of m_c^0/m_t^0 . Curiously, owing to the rank-2 nature of Δ_{u^c} , this is not so. Generically, with the elements of $\Delta_{u^c} = O(\frac{1}{\epsilon^2})$ and given by eq.(20), $(I + \Delta_{u^c})^{-\frac{1}{2}}$ suppresses m_c^0/m_t^0 by a factor $O(\epsilon)$. More precisely, if $|\vec{E}| \sim |\vec{E}'| > \epsilon$, then $(m_c^0/m_t^0)/(m_s^0/m_b^0) \cong \frac{4}{5}\epsilon \left| \frac{f\Omega}{\langle \mathbf{1}(\mathbf{16}_H) \rangle} \right| \frac{|\vec{E} \times \vec{E}'| [E_1^2 + E_1'^2]^{\frac{1}{2}}}{[(\vec{E} \times \vec{E}')_3]^2} = O(\epsilon)$.

It is assumed that while $\langle \mathbf{1}(\mathbf{16}_H) \rangle$ is non-zero and superlarge, $\langle \mathbf{5}(\mathbf{16}_H) \rangle$ vanishes. This condition can be naturally achieved¹⁶ and means that $W_{adjoint}$ will contribute no antisymmetric piece to U_0 . The only effect, then, of $W_{adjoint}$ is the suppression of m_c^0/m_t^0 . All the successful relations **(I)** – **(IX)** remain essentially untouched. In particular, the effect of $(I + \Delta_{u^c})^{-\frac{1}{2}}$ on V_{cb}^0 is to make $V_{cb}^0 \cong \frac{2}{5}\epsilon \sin \theta \cos \theta \left[1 + \frac{N}{2} \tan \theta \frac{(\vec{E} \times \vec{E}')_2}{(\vec{E} \times \vec{E}')_3} \right]$. This version of the model has the following predictions.

$$\begin{aligned} \tan \beta &\cong v/v' \cong m_t^0/m_b^0, \\ \theta_{e\mu} &\cong \sqrt{m_e^0/m_\mu^0} + O(m_u^0/m_c^0), \\ \theta_{e\tau} &\cong V_{ub} + O(m_u^0/m_t^0). \end{aligned} \tag{21}$$

$\theta_{\mu\tau}$ gets a large contribution from $L_{32}/L_{33} \cong \tan \theta \sim 1$, and can therefore be very large. But for a wide range of the parameters describing M_R , the Majorana mass matrix of the right-handed neutrinos, $\theta_{\mu\tau} \cong 3V_{cb}$ (see Ref. (17)) due to the near cancellation between L_{32}/L_{33} and N_{32}/N_{33} . More precisely, for $(M_R^{-1})_{22} \lesssim (M_R^{-1})_{23} \lesssim (M_R^{-1})_{33}$, $\theta_{\mu\tau} \cong 3V_{cb}(1 + \frac{1}{2}[(M_R^{-1})_{23}/(M_R^{-1})_{33}]N \tan \theta)$. It should be noted, finally, that in this version of the model, θ comes out to be of order 1, so that there is no unexplained alignment in family space, and ϵ comes out of order $\frac{1}{10}$. (This can be seen from the relation $m_s^0/m_b^0 \cong \frac{N}{5}\epsilon \sin^2 \theta$ and the relation just given for V_{cb}^0 .)

In conclusion, we have shown that the group theory of $SO(10)$ can elegantly explain the pattern of fermion masses and mixing angles without family symmetry. We find it remarkable that the nine features listed in the introduction can arise as a consequence of the simple Yukawa terms of eq.(1). We also find it remarkable that the single group-theoretical assumption $Q \sim I_{3R}$ explains three of those relations as well as the smallness of m_c^0/m_t^0 . The fact that the gauge group is $SO(10)$ has played several crucial roles in the model: it relates the up quarks and neutrinos to the down quarks and leptons, it allows the VEVs of the adjoint Higgs fields to point in the I_{3R} and X directions, and it makes possible the antisymmetric contributions to the mass matrices coming from W_{vector} . We have only discussed terms in the superpotential that are directly relevant to understanding the pattern of light fermion masses. Other terms will be present including the Higgs sector¹² and additional small Yukawa couplings¹⁸, but our results are not sensitive to these. Details of the numerical fits and certain technical points will be presented in a longer paper.

References

1. S. Dimopoulos, L. Hall and S. Raby, Phys. Rev. Lett. **68**, 752 (1992); H. Arason, D. Castano, E. Pirad and P. Ramond, Phys. Rev. **D 47**, 232 (1992); for a comprehensive review see S. Raby, OSU preprint OHSTPY-HEP-T-95-024, hep-ph/9501349.
2. None of the discussions in this paper depends on supersymmetry, although it will be assumed.
3. A. Buras, J. Ellis, M.K. Gaillard and D.V. Nanopoulos, Nucl. Phys. **B135**, 66 (1978).
4. H. Georgi and C. Jarlskog, Phys. Lett. **86B**, 297 (1979).
5. S. Weinberg, in *Festschrift for I.I. Rabi*, Trans. N.Y. Acad. Sci. Ser. II **38**, 185 (1977); F. Wilczek and A. Zee, Phys. Lett. **70B**, 418 (1977); H. Fritzsch, *ibid.*, **70B**, 436 (1977); R. Gatto, G. Sartori and M. Tonin, *ibid.*, **28B**, 128 (1968); R.J. Oakes, *ibid.*, **29B**, 683 (1969).
6. This form can be enforced by a discrete Z_3 symmetry.
7. S.M. Barr, Phys. Rev. **D 24**, 1895 (1981); B.S. Balakrishna, A.L. Kagan and R.N. Mohapatra, Phys. Lett. **B205**, 345 (1988); Z. Berezhiani and R. Rattazzi, Nucl. Phys. **B 407**, 249 (1993); K.S. Babu and E. Ma, Mod. Phys. Lett. **A4**, 1975 (1989) and references therein.
8. S.M. Barr, Phys. Rev. Lett. **64**, 353 (1990).

9. More precisely Q is in general a complex linear combination of generators of $SO(10)$.
10. K.S. Babu and S.M. Barr, BA-94-45, hep-ph/9409285.
11. M. Srednicki, Nucl. Phys. **B202**, 327 (1982).
12. S. Dimopoulos and F. Wilczek, Report No. NSF-ITP-82-07, August 1981 (unpublished); K.S. Babu and S.M. Barr, Phys. Rev. **D48**, 5354 (1993); Phys. Rev. **D50**, 3529 (1994).
13. H. Fritzsch, Nucl. Phys. **B155**, 189 (1979).
14. All parameters (eg. $\epsilon, c_i, c'_i, e_i, e'_i$) are assumed to be complex. With their phases of order unity sufficient KM CP violation results.
15. $\tan\beta \cong v/v'$ holds only if the mixing of $\bar{\mathbf{5}}(\mathbf{16}_H)$ with the $\bar{\mathbf{5}}(\mathbf{10}_H)$ is small. The smallness of this mixing can naturally explain why the $c_{ij} \ll 1$. For a discussion of gauge hierarchy including such mixing, see Ref. (16).
16. K.S. Babu and R.N. Mohapatra, Preprint BA-94-56, UMD-PP-95-57, hep-ph/9410326.
17. A similar relation for $\theta_{\mu\tau}$ is derived in K.S. Babu and R.N. Mohapatra, Phys. Rev. Lett **70**, 2845 (1993); but the prediction for $\theta_{e\tau}$ is different.
18. For example, small $\mathbf{45}^2$ and $\mathbf{45}'^2$ mass terms may be added which will make all components of $\mathbf{45}, \mathbf{45}'$ superheavy and generate large ν_i^c Majorana masses.